Problem 1. Show that if $Z$is normal(0,1) then $\mathbb{E}e^{aZ}\mathbb{I}(Z > b) = e^{\frac{1}{2}a^2}N(a-b)$

$$\mathbb{E}e^{aZ}\mathbb{I}(Z > b) = e^{\frac{1}{2}a^2}\mathbb{E}\mathbb{I}(Z + a > b) = e^{\frac{1}{2}a^2}\mathbb{E}\mathbb{I}(-Z + a > b) = e^{\frac{1}{2}a^2}N(a-b)$$

Problem 2. Show that if $a, b, \sigma > 0$ are constant then $\mathbb{E}\left(ae^{\sigma Z - \frac{1}{2}\sigma^2} - b\right)^+ = aN(d_1) - bN(d_{-1})$

$$\begin{aligned}
\mathbb{E}\left(ae^{\sigma Z-\frac{1}{2}\sigma^2} - b\right)^+ &= \mathbb{E}\left(ae^{\sigma Z-\frac{1}{2}\sigma^2} - b\right)\mathbb{I}(ae^{\sigma Z-\frac{1}{2}\sigma^2} > b) \\
&= a\mathbb{E}\mathbb{I}(ae^{\sigma(Z+\sigma)-\frac{1}{2}\sigma^2} > b) - b\mathbb{E}\mathbb{I}(ae^{\sigma Z-\frac{1}{2}\sigma^2} > b) \\
&= a\mathbb{E}\mathbb{I}(Z > \sigma^{-1}\ln\frac{b}{a} - \frac{1}{2}\sigma) - b\mathbb{E}\mathbb{I}(Z > \sigma^{-1}\ln\frac{b}{a} + \frac{1}{2}\sigma) \\
&= a\mathbb{E}\mathbb{I}(Z < \sigma^{-1}\ln\frac{a}{b} + \frac{1}{2}\sigma) - b\mathbb{E}\mathbb{I}(Z < \sigma^{-1}\ln\frac{a}{b} - \frac{1}{2}\sigma) \\
&= aN(d_1) - bN(d_{-1})
\end{aligned}$$

Problem 3. Compute the MGF of a normal(μ, σ^2) random variabl

$$\phi_Y(t) = \mathbb{E}e^{tY} = \mathbb{E}e^{t(\mu+\sigma Z)} = e^{\mu t}\mathbb{E}e^{t\sigma Z} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Problem 4. Prove that the MGF of the sum of two independent random variables is equal to the product of their MGFs

$$\phi_{X+Y}(t) = \mathbb{E}e^{t(X+Y)} = \mathbb{E}e^{tX}e^{tY} = \mathbb{E}e^{tX}\mathbb{E}e^{tY} = \phi_X(t)\phi_Y(t)$$

where the property that independence is preserved under transformation by Borel functions, in this case $f(x) = e^{tx}$

Problem 5. Use the prior result to show that the sum of independent normals is a normal

$$\phi_{\sum_i c_i X_i}(t) = \prod_i \phi_{c_i X_i}(t) = \prod_i e^{c_i\mu_i t + \frac{1}{2}c_i^2\sigma_i^2 t^2} = e^{\sum_i c_i\mu_i t + \frac{1}{2}\sum_i c_i^2\sigma_i^2 t^2}$$

which is the MGF of a normal$(\sum_i c_i\mu_i, \sum_i c_i^2\sigma_i^2)$ random variable.

Problem 6. Let $\mathcal{L}(y|\mu, \sigma^2)$ denote the CDF of the log-normal distribution. Derive the following results:

1. $\mu_k = \mathbb{E}Y^k = e^{k\mu + \frac{1}{2}k^2\sigma^2}$
2. $\mathbb{E}Y^k\mathbb{I}(Y < x) = \mu_k\mathcal{L}(x|\mu + k\sigma^2, \sigma^2)$
3. $\mathbb{E}Y^k\mathbb{I}(Y > x) = \mu_k\mathcal{L}(x^{-1}| - (\mu + k\sigma^2), \sigma^2)$

Since $\ln Y$ is a normal(μ, σ^2) random variable, $Y = e^{\mu+\sigma Z}$ where $Z$is normal(0,1). Hence $Y^k = e^{k\mu+k\sigma Z}$ and so by the GSL, $\mu_k = e^{k\mu+\frac{1}{2}k^2\sigma^2}$. Also

$$\begin{aligned}
\mathbb{E}Y^k\mathbb{I}(Y < x) &= \mathbb{E}e^{k\mu+k\sigma Z}\mathbb{I}(e^{\mu+\sigma Z} < x) \\
&= \mu_k\mathbb{E}\mathbb{I}(e^{\mu+\sigma(Z+k\sigma)} < x) \\
&= \mu_k\mathbb{E}\mathbb{I}(e^{\mu+k\sigma^2+\sigma Z} < x) \\
&= \mu_k\mathcal{L}(x|\mu + k\sigma^2, \sigma^2)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}Y^k\mathbb{I}(Y > x) &= \mathbb{E}e^{k\mu+k\sigma Z}\mathbb{I}(e^{\mu+\sigma Z} > x) \\
&= \mathbb{E}e^{k\mu+k\sigma Z}\mathbb{I}(e^{-\mu-\sigma Z} < x^{-1}) \\
&= \mu_k\mathbb{E}\mathbb{I}(e^{-\mu-\sigma(Z+k\sigma)} < x^{-1}) \\
&= \mu_k\mathcal{L}(x^{-1}| - (\mu + k\sigma^2), \sigma^2)
\end{aligned}$$

and so when $k = 0$, $\mathcal{L}(x|\mu, \sigma^2) + \mathcal{L}(x^{-1}| - \mu, \sigma^2) = 1$

Problem 7. Let $N(x, y; \rho)$ denote the bivariate normal CDF. Show that

$$\mathbb{E}\mathbb{I}(X < b)N\left(\frac{a - \rho X}{(1 - \rho^2)^{1/2}}\right) = N(a, b; \rho)$$

Note that

$$\begin{aligned}
N(a, b; \rho) &= \mathbb{E}\mathbb{I}(Y < a, X < b) \\
&= \mathbb{E}\mathbb{E}_X\mathbb{I}(Y < a, X < b) \\
&= \mathbb{E}\mathbb{I}(X < b)\mathbb{E}_X\mathbb{I}(Y < a) \\
&= \mathbb{E}\mathbb{I}(X < b)\mathbb{P}[Y < a|X] \\
&= \mathbb{E}\mathbb{I}(X < b)\mathbb{P}\left[\frac{Y - \rho X}{\beta} < \frac{a - \rho X}{\beta}|X\right] \\
&= \mathbb{E}\mathbb{I}(X < b)N\left(\frac{a - \rho X}{\beta}\right)
\end{aligned}$$

where the property that $Y|X$ is normal$(\rho X, \beta^2)$

Problem 8. Use prior problem to show that

1. $N(x, y; 0) = N(x)N(y)$

2. $N(x, y; 1) = N(x \wedge y)$

3. $N(x, y; -1) = (N(x) - N(-y))\mathbb{I}(x + y > 0)$

Since $N(x, y; 0) = \mathbb{E}\mathbb{I}(X < x)N(y) = N(x)N(y)$ first item is true. Since $N(x, y; 1) = \lim_{\rho \uparrow 1} \mathbb{E}\mathbb{I}(X < x)N\left(\frac{y-\rho X}{\beta}\right) = \mathbb{E}\mathbb{I}(X < x)\mathbb{I}(y > X) = \mathbb{E}\mathbb{I}(X < x \wedge y) = N(x \wedge y)$ the second item is true. Finally,

$$
\begin{aligned}
N(x, y; -1) &= \lim_{\rho \downarrow 1} \mathbb{E}\mathbb{I}(X < x)N\left(\frac{y - \rho X}{\beta}\right) \\
&= \mathbb{E}\mathbb{I}(X < x)\mathbb{I}(y + X > 0) \\
&= \mathbb{E}\mathbb{I}(-y < X < x) \\
&= [N(x) - N(-y)]\mathbb{I}(-y < x)
\end{aligned}
$$

Problem 9. If X_i are correlated normal(μ_i, σ_i^2) random variables with covariance matrix $\Gamma_{ij} = \sigma_i \sigma_j \rho_{ij}$ show that $Y = \sum_i c_i X_i$ is univariate normal$(c^T \mu, c^T \Gamma c)$. Since $X = \mu + \Gamma^{1/2} Z$ where $Z$is an independent random vector whose components are normal(0,1),

$$
\begin{aligned}
\phi_Y(t) &= \mathbb{E}e^{tc^T X} \\
&= \mathbb{E}e^{tc^T(\mu + \Gamma^{1/2} Z)} \\
&= e^{c^T \mu \cdot t}\mathbb{E}e^{c^T \Gamma^{1/2} Z} \\
&= e^{c^T \mu \cdot t + \frac{1}{2} c^T \Gamma^{1/2} \Gamma^{1/2} c}
\end{aligned}
$$

where the symmetry of the covariance matrix is used.

Problem 10. If X, Y are $N(0, 1; \rho)$ show that $\mathbb{E}Y f(X) = \rho \mathbb{E} f'(X)$ where f is a given differentiable and measurable function.

$$
\begin{aligned}
\mathbb{E}Y f(X) &= \mathbb{E}\mathbb{E}_X Y f(X) \\
&= \mathbb{E} f(X) \mathbb{E}_X Y \\
&= \rho \mathbb{E} f(X) X \\
&=? \ \rho \mathbb{E} f'(X)
\end{aligned}
$$

Problem 11. Let X, Y satisfy GBM SDEs:

$$
\begin{aligned}
dX_t &= X_t(\mu_x dt + \sigma_x dB_t) \\
dY_t &= Y_t(\mu_y dt + \sigma_y dB_t)
\end{aligned}
$$

Derive the product and quotient SDEs.

$$
\begin{aligned}
d(X_tY_t) &= Y_t dX_t + X_t dY_t + dX_t dY_t \\
&= X_tY_t\left(\frac{dX_t}{X_t}+\frac{dY_t}{Y_t}+\frac{dX_t}{X_t}\frac{dY_t}{Y_t}\right) \\
&= X_tY_t\left((\mu_x+\mu_y+\sigma_x\sigma_y)dt+(\sigma_x+\sigma_y)dB_t\right)
\end{aligned}
$$

and

$$
\begin{aligned}
d(X_tY_t^{-1}) &= dX_tY_t^{-1} - X_tY_t^{-2}dY_t - Y_t^{-2}dX_tdY_t + X_tY_t^{-3}dY_t^2 \\
&= X_tY_t^{-1}\left(\frac{dX_t}{X_t}-\frac{dY_t}{Y_t}-\frac{dX_t}{X_t}\frac{dY_t}{Y_t}+\frac{dY_t}{Y_t}\frac{dY_t}{Y_t}\right) \\
&= X_tY_t^{-1}\left((\mu_x-\mu_y-\sigma_x\sigma_y+\sigma_y^2)dt+(\sigma_x-\sigma_y)dB_t\right)
\end{aligned}
$$

Problem 12. Show that $\mathbb{E}\left(ae^{\sigma_1X-\frac{1}{2}\sigma_1^2}-be^{\sigma_2Y-\frac{1}{2}\sigma_2^2}\right)^+ = aN(d_1)-bN(d_2)$

$$
\begin{aligned}
\mathbb{E}\left(ae^{\sigma_1X-\frac{1}{2}\sigma_1^2}-be^{\sigma_2Y-\frac{1}{2}\sigma_2^2}\right)^+ &= \mathbb{E}\left(ae^{\sigma_1X-\frac{1}{2}\sigma_1^2}-be^{\sigma_2Y-\frac{1}{2}\sigma_2^2}\right)\mathbb{I}(ae^{\sigma_1X-\frac{1}{2}\sigma_1^2}>be^{\sigma_2Y-\frac{1}{2}\sigma_2^2}) \\
&= a\mathbb{E}\mathbb{I}(ae^{\sigma_1(X+\sigma_1)-\frac{1}{2}\sigma_1^2}>be^{\sigma_2(Y+\rho\sigma_1)-\frac{1}{2}\sigma_2^2}) - b\mathbb{E}\mathbb{I}(ae^{\sigma_1(X+\rho\sigma_2)-\frac{1}{2}\sigma_1^2}>be^{\sigma_2(Y+\sigma_2)-\frac{1}{2}\sigma_2^2}) \\
&= a\mathbb{E}\mathbb{I}\left(\ln\frac{a}{b}+\frac{1}{2}\left(\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2\right)>\sigma_2Y-\sigma_1X\right) \\
&\quad -b\mathbb{E}\mathbb{I}\left(\ln\frac{a}{b}-\frac{1}{2}\left(\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2\right)>\sigma_2Y-\sigma_1X\right) \\
&= aN\left(\frac{1}{\sigma}\ln\frac{a}{b}+\frac{1}{2}\sigma\right)-bN\left(\frac{1}{\sigma}\ln\frac{a}{b}-\frac{1}{2}\sigma\right)
\end{aligned}
$$

Since $Z=\sigma_2Y-\sigma_1X$ is normal(0,$\sigma \triangleq \sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2$)

Problem 13. If $Z$is normal(0,1) show that $\mathbb{E}N(a_1Z+b_1)N(a_2Z+b_2)=N(c_1,c_2;\rho)$

$$
\begin{aligned}
\mathbb{P}_ZN(a_1Z+b_1)N(a_2Z+b_2) &= \mathbb{E}\mathbb{E}\mathbb{I}(U<a_1Z+b_1)\mathbb{E}\mathbb{I}(V<a_2Z+b_2) \\
&= \mathbb{E}\mathbb{E}\mathbb{I}\left(\frac{U-a_1Z}{(1-a_1^2)^{1/2}}<\frac{b_1}{(1-a_1^2)^{1/2}}\right)\mathbb{I}\left(\frac{V-a_2Z}{(1-a_2^2)^{1/2}}<\frac{b_2}{(1-a_2^2)^{1/2}}\right) \\
&= N\left(\frac{b_1}{(1-a_1^2)^{1/2}},\frac{b_2}{(1-a_2^2)^{1/2}};\frac{a_1a_2}{(1-a_1^2)^{1/2}(1-a_2^2)^{1/2}}\right)
\end{aligned}
$$

Problem 14. Show that $N(a,b;\rho) = N(a,c;\rho_1) + N(b,-c;\rho_2)$. Let X, Y be normal(0,1;ρ). Let $y = \alpha x + \beta$ be a line passing through (a,b). Then $\beta = b - \alpha a$

$$\begin{aligned}
N(a,b;\rho) &= \mathbb{E}\mathbb{I}(X < a, Y < b) \\
&= \mathbb{E}\mathbb{I}(X < a, Y < b, Y < \alpha X + \beta) + \mathbb{E}\mathbb{I}(X < a, Y < b, Y > \alpha X + \beta) \\
&= \mathbb{E}\mathbb{I}(X < \alpha^{-1}(Y - \beta), Y < b) + \mathbb{E}\mathbb{I}(X < a, Y < \alpha X + \beta) \\
&= \mathbb{E}\mathbb{I}\left(\alpha X - Y < -\beta, Y < b\right) + \mathbb{E}\mathbb{I}(X < a, Y - \alpha X < \beta) \\
&= N\left(\frac{-\beta}{(1-2\rho\alpha+\alpha^2)^{1/2}}, b; -\frac{1}{(1-2\rho\alpha+\alpha^2)^{1/2}}\right) + N\left(a, \frac{\beta}{(1-2\rho\alpha+\alpha^2)^{1/2}}; \frac{-\alpha}{(1-2\rho\alpha+\alpha^2)^{1/2}}\right)
\end{aligned}$$

If $X$is chosen to have variance σ_1^2 and Y is chosen to have variance σ_2^2 and the slope $\alpha = 1$, then $\beta = \sigma_2 b - \sigma_1 a$ and the above analysis can be repeated to attain the desired conclusion of Büchen q10:

$$\begin{aligned}
N(a,b;\rho) &= \mathbb{E}\mathbb{I}(X < a\sigma_1, Y < b\sigma_2) \\
&= \mathbb{E}\mathbb{I}(X - Y < -\beta, Y < b\sigma_2) + \mathbb{E}\mathbb{I}(X < a\sigma_1, Y - X < \beta) \\
&= N\left(\frac{-\beta}{\sigma}, \frac{b\sigma_2}{\sigma_2}; \frac{\rho\sigma_1\sigma_2 - \sigma_2^2}{\sigma\sigma_2}\right) + N\left(\frac{a\sigma_1}{\sigma_1}, \frac{\beta}{\sigma}; \frac{\rho\sigma_1\sigma_2 - \sigma_1^2}{\sigma\sigma_1}\right) \\
&= N\left(\frac{-\beta}{\sigma}, b; \frac{\rho\sigma_1 - \sigma_2}{\sigma}\right) + N\left(a, \frac{\beta}{\sigma}; \frac{\rho\sigma_2 - \sigma_1}{\sigma}\right)
\end{aligned}$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$

Problem 15. Show that $N(a,b;\rho) = N(a,c;-\beta) + N(b)N(-c)$ where $c = \frac{\rho b - a}{\beta}, \beta = (1-\rho^2)^{1/2}$

From problem 14, choose $\sigma_2 = \rho\sigma_1$ to zero the correlation of one term, and choose $\beta = c\sigma$. Then $N(a,b;\rho) = N(-c,b;0) + N(a,c,-\frac{(1-\rho^2)\sigma_1}{(\sigma_1^2+\rho^2\sigma_1^2-2\rho^2\sigma_1^2)^{1/2}} = (1-\rho^2)^{1/2})$. Since uncorrelated normal random variables are independent, $N(-c,b;0) = N(-c)N(b)$ and the conclusion follows.