1 Estimating Mean and Variance

Given an iid sequence of random variables with unknown mean μ and variance σ^2 the following estimators are unbiased:

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
\hat{\sigma^2} &= \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{1}{n}\sum_{i=1}^{n}X_i^2 - 2\mu\sum_{i=1}^{n}X_i + \mu^2
\end{aligned}
$$

Note that

$$
\mathrm{var}\hat{\mu} = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}X_i = \frac{1}{n}\sigma^2
$$

The second estimator requires knowledge of the mean. Consider the statistic $T = \sum_{i=1}^{n}(X_i - \hat{\mu})^2$

$$
\begin{aligned}
\mathbb{E}T &= \mathbb{E}\sum_{i=1}^{n}(X_i - \mu + \mu - \hat{\mu})^2 \\
&= \mathbb{E}\sum_{i=1}^{n}\big((X_i-\mu)^2 + 2(X_i-\mu)(\mu-\hat{\mu}) + (\mu-\hat{\mu})^2\big) \\
&= n\sigma^2 - 2\sum_{i=1}^{n}\mathrm{cov}(X_i-\mu, \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu) + n\mathrm{var}\hat{\mu} \\
&= n\sigma^2 - 2\sum_{i=1}^{n}\frac{1}{n}\sigma^2 + \sigma^2 \\
&= (n-1)\sigma^2
\end{aligned}
$$

And so $\hat{\sigma} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$ is an unbiased estimator. By the LLN these estimators converge almost surely as the number of sample increases without bound.